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The first integrals and orbit equation for the Kepler problem with drag

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Abstract. First integrals for the classical Kepler problem with drag were obtained by Jezewski and Mittleman. A derivation which is more attractive from an intuitive point of view is provided and this leads naturally to the orbit equation as for the standard Kepler problem.

1. Introduction

The Kepler problem with drag has been of some interest in the study of the motion of satellites moving sufficiently close to the Earth to be affected by its atmosphere. Few solutions, either analytic or in closed form, exist. Brouwer and Hori (1961) obtained a closed form solution which included first-order corrections due to a velocity square law in drag acceleration. Danby (1962) proposed a drag law which was proportional to the velocity and inversely proportional to the square of the radial distance. Assuming that the constant of proportionality was small he obtained a first-order perturbation solution. Mittleman and Jezewski (1982) obtained an analytic solution of this same problem and, in Jezewski and Mittleman (1983), demonstrated that there existed first integrals which were the direct analogues of the angular momentum, the energy and the Laplace-Runge-Lenz vector of the classical Kepler problem.

The method which they adopted was the simple one of manipulating the equation of motion as had been used by Collinson (1973) in his study of the Kepler problem and Sarlet and Bahar (1980) for a variety of non-linear problems. Perhaps because of the influence of earlier work on the Kepler problem with drag the actual derivation of the integrals was not as simple as the principle used and the physical intuition inherent in the approach tended to become lost in the mathematics.

In this paper we show how the first integrals and orbit equation for the Kepler problem with the drag law proposed by Danby and Mittleman and Jezewski can be obtained in a simple fashion which emphasises the physics and does not become obscured by unnecessarily complicated mathematics. We do not suggest that this simple approach is the best method to use in the search for first integrals of classical motions. It is specific and should not be compared with general methods such as Noether's theorem (for an excellent review of which see Sarlet and Cantrijn (1981)) or the 'direct approach' (Gascón *et al* 1982, Lewis and Leach 1982, Moreira 1983). However, for this problem the form of the Lagrangian is not yet known and so it is

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not possible to use Noether's theorem and the first integrals are such that the use of the direct approach would require considerable inventiveness in guessing an appropriate ansatz. The Lie theory of extended groups was used by Leach (1981) to obtain the first integrals of the Kepler problem, but it has not been possible (thus far) to do the same when drag is present. We have remarked that the method to be used here is specific, but it does appear to be suited to problems related to the Kepler problem. Recently Gorringe and Leach (1986) used it to obtain the first integrals for the 'time-dependent Kepler problem' for which the equation of motion is

$$\ddot{\boldsymbol{r}} = \ddot{u}(t)\boldsymbol{r} + \mu\boldsymbol{r}/r^3u(t) \tag{1}$$

where $u(t)$ is an arbitrary (twice differentiable) function of time. This problem had been previously treated with more standard approaches by Katzin and Levine (1983) and Leach (1985). Consequently, despite the specific nature of the method, we suggest that it may be fruitful to investigate just how wide its area of applicability is.

2. An integral related to angular momentum

The classical Kepler problem with drag proportional to the velocity and inversely proportional to the square of the radial distance may be described by the equation of motion

$$\ddot{\boldsymbol{r}} + \frac{\alpha\dot{\boldsymbol{r}}}{r^2} + \frac{\mu\boldsymbol{r}}{r^3} = 0 \tag{2}$$

in reduced coordinates where α and μ are constants. (In fact one can rescale the variables to make α and μ both one, but they are kept for physical considerations.) Taking the vector product of (2) with $\boldsymbol{r}$ we have

$$\begin{aligned} \boldsymbol{r}\times\ddot{\boldsymbol{r}} + \alpha\frac{\boldsymbol{r}\times\dot{\boldsymbol{r}}}{r^2} &= 0 \\ \dot{\boldsymbol{L}} + \frac{\alpha\boldsymbol{L}}{r^2} &= 0 \end{aligned} \tag{3}$$

where $\boldsymbol{L} = \boldsymbol{r}\times\dot{\boldsymbol{r}}$ is the angular momentum. Clearly the angular momentum is not conserved, but, on taking the vector product of (3) with $\boldsymbol{L}$, we see that

$$\dot{\boldsymbol{L}}\times\boldsymbol{L} = 0. \tag{4}$$

Hence $\boldsymbol{L}$ is parallel to $\dot{\boldsymbol{L}}$ and so the unit vector in the direction of $\boldsymbol{L}$, $\hat{\boldsymbol{L}}$, is constant (here, as elsewhere $\hat{}$ is used to denote a unit vector). The motion lies in a fixed plane and we take the origin to lie within the plane so that, when we require a coordinate representation, we may use plane polar coordinates (r, θ). We rewrite (3) as

$$(\dot{L} + \alpha\dot{\theta})\hat{\boldsymbol{L}} = 0 \tag{5}$$

from which it is evident that there exists a conserved scalar

$$h = L + \alpha\theta. \tag{6}$$

3. The generalisation of Hamilton's vector

Equation (3) contains still more useful information. Rewriting it as

$$\frac{\alpha}{r^2}=-\frac{\dot{L}}{L} \tag{7}$$

our equation of motion (2) takes the form

$$\ddot{\boldsymbol{r}}-\frac{\dot{L}}{L}\dot{\boldsymbol{r}}+\frac{\mu\boldsymbol{r}}{r^3}=0. \tag{8}$$

Dividing by L we have

$$0=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\boldsymbol{r}}}{L}\right)+\frac{\mu\boldsymbol{r}}{r^3L}. \tag{9}$$

Recognising that $r^2\dot{\theta}=L=h-\alpha\theta$ it follows that there exists a vector first integral which is the appropriate generalisation of Hamilton's vector for the Kepler problem. It is

$$\boldsymbol{K}=\frac{\dot{\boldsymbol{r}}}{L}+\mu\int\frac{\hat{\boldsymbol{r}}\dot{\theta}}{(h-\alpha\theta)^2}\,\mathrm{d}t \tag{10}$$

in which the integral is evaluated in terms of si and ci functions as follows. Writing $\hat{\boldsymbol{r}}$ as $\boldsymbol{i}\cos\theta+\boldsymbol{j}\sin\theta$ we have (see Gradshteyn and Ryzhik (1965) 2.641.1-4)

$$\int\frac{\cos\theta\,\mathrm{d}\theta}{(h-\alpha\theta)^2}=\frac{1}{\alpha}\frac{\cos\theta}{h-\alpha\theta}-\frac{1}{\alpha^2}\left[\cos\frac{h}{\alpha}\,\mathrm{si}\left(\theta-\frac{h}{\alpha}\right)+\sin\frac{h}{\alpha}\,\mathrm{ci}\left(\theta-\frac{h}{\alpha}\right)\right] \tag{11}$$

$$\int\frac{\sin\theta\,\mathrm{d}\theta}{(h-\alpha\theta)^2}=\frac{1}{\alpha}\frac{\sin\theta}{h-\alpha\theta}+\frac{1}{\alpha^2}\left[\cos\frac{h}{\alpha}\,\mathrm{ci}\left(\theta-\frac{h}{\alpha}\right)-\sin\frac{h}{\alpha}\,\mathrm{si}\left(\theta-\frac{h}{\alpha}\right)\right] \tag{12}$$

so that the conserved vector is

$$\begin{aligned}\boldsymbol{K}=\frac{\dot{\boldsymbol{r}}}{L}+\frac{\mu}{\alpha}\frac{\hat{\boldsymbol{r}}}{h\alpha\theta}+\frac{\mu}{\alpha^2}\bigg\{&\left[\sin\left(\theta-\frac{h}{\alpha}\right)\mathrm{ci}\left(\theta-\frac{h}{\alpha}\right)-\cos\left(\theta-\frac{h}{\alpha}\right)\mathrm{si}\left(\theta-\frac{h}{\alpha}\right)\right]\hat{\boldsymbol{r}}\\&+\left[\cos\left(\theta-\frac{h}{\alpha}\right)\mathrm{ci}\left(\theta-\frac{h}{\alpha}\right)+\sin\left(\theta-\frac{h}{\alpha}\right)\mathrm{si}\left(\theta-\frac{h}{\alpha}\right)\right]\hat{\boldsymbol{\theta}}\bigg\}.\end{aligned} \tag{13}$$

It is evident that the analogue of the Laplace-Runge-Lenz vector is

$$\begin{aligned}\boldsymbol{J}=\boldsymbol{K}\times\hat{\boldsymbol{L}}=&\frac{\dot{\boldsymbol{r}}\times\hat{\boldsymbol{L}}}{L}-\frac{\mu}{\alpha}\frac{\hat{\theta}}{h-\alpha\theta}\\&+\frac{\mu}{\alpha^2}\bigg\{\left[\cos\left(\theta-\frac{h}{\alpha}\right)\mathrm{ci}\left(\theta-\frac{h}{\alpha}\right)+\sin\left(\theta-\frac{h}{\alpha}\right)\mathrm{si}\left(\theta-\frac{h}{\alpha}\right)\right]\hat{\boldsymbol{r}}\\&-\left[\sin\left(\theta-\frac{h}{\alpha}\right)\mathrm{ci}\left(\theta-\frac{h}{\alpha}\right)-\cos\left(\theta-\frac{h}{\alpha}\right)\mathrm{si}\left(\theta-\frac{h}{\alpha}\right)\right]\hat{\boldsymbol{\theta}}\bigg\}.\end{aligned} \tag{14}$$

These expressions for the first integrals may be written more compactly if we define

$$g(\theta)=\frac{\mu}{\alpha^2}\left[\cos\left(\theta-\frac{h}{\alpha}\right)\mathrm{ci}\left(\theta-\frac{h}{\alpha}\right)+\sin\left(\theta-\frac{h}{\alpha}\right)\mathrm{si}\left(\theta-\frac{h}{\alpha}\right)\right] \tag{15}$$

so that now

$$K = \frac{\dot{r}}{L} - g'(\theta)\hat{r} + g(\theta)\hat{\theta} \tag{16}$$

$$J = \frac{\dot{r} \times \hat{L}}{L} + g(\theta)\hat{r} + g'(\theta)\hat{\theta}. \tag{17}$$

4. An energy-like integral

The existence of the two conserved vectors is sufficient to specify the motion. Typically one searches for an energy-like first integral even when it is not necessary. For example in the case of the Kepler problem we find that energy, angular momentum and Hamilton's vector $e(=\dot{r} + \mu\hat{r}/L)$ are related according to

$$E = \tfrac{1}{2}e \cdot e - \tfrac{1}{2}\mu^2/L^2. \tag{18}$$

Such a relationship does not persist in the case of the Kepler problem with drag. There is a small mistake in Jezewski and Mittleman (1983) where it is stated that the energy-like integral is (in our notation) $J \cdot J$ to within an additive constant. In fact it is $\frac{1}{2}J \cdot J$.

The reason for this departure from the classical Kepler problem is found in the problem of integration. For the Kepler problem with

$$\ddot{r} + \frac{\mu r}{r^3} = 0 \tag{19}$$

it suffices to take the scalar product with $\dot{r}$ as this then renders the second term as an exact derivative. In the present problem the relevant equation is (9) and there is a term L^{-1} in the second term and this is no longer constant as in the Kepler problem. We must rewrite (9) as

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{r}}{L}\right) + \frac{\mu\hat{r}\dot{\theta}}{L^2} = \dot{K} \tag{20}$$

and take the scalar product with K to obtain the energy-like first integral

$$I = \frac{1}{2}\frac{\dot{r} \cdot \dot{r}}{L^2} - \frac{1}{L}(\dot{r}g'(\theta) + r\dot{\theta}g(\theta)) + \tfrac{1}{2}(g'(\theta)^2 + g(\theta)^2). \tag{21}$$

However, for the reason which we outlined above we do not become as excited about I as is our wont for other problems. Were one to somehow come up with a quantum mechanical Coulomb problem with such a drag law, I would probably be of use, but its interpretation as an operator could pose difficulties due to the inverse powers of L.

5. The orbit equation

As in other Kepler-like problems the analogue of the Laplace-Runge-Lenz vector plays a key role in the determination of the orbit equation. If we now specify θ to be

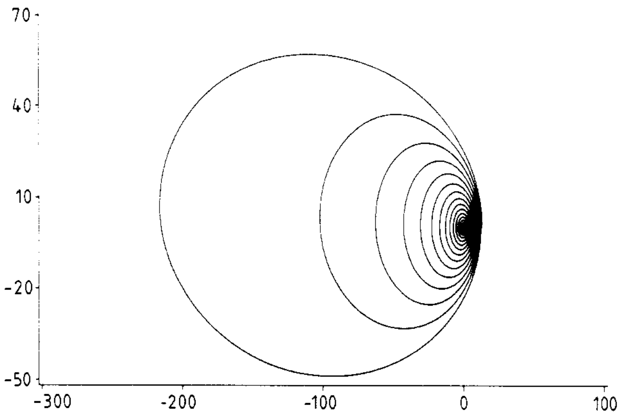

Figure 1. Orbit for $J = \frac{1}{13}$, $\mu = 500$, $h = 36\pi$, $\alpha = 1$.

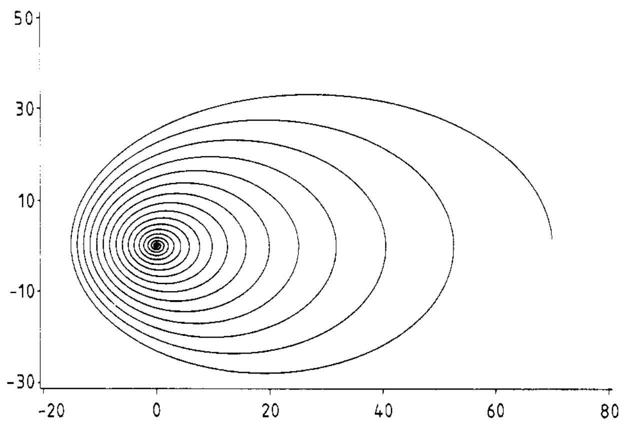

Figure 2. Orbit for $J = \frac{1}{70}$, $\mu = 500$, $h = 36\pi$, $\alpha = 1$.

the angle between $\boldsymbol{J}$ and $\boldsymbol{r}$ and take the scalar product of $\boldsymbol{J}$ with $\boldsymbol{r}$, we have, after some rearrangement,

$$r(\theta) = \frac{1}{J \cos\theta - g(\theta)}. \tag{22}$$

We may use this relationship to compute orbits for various values of the parameters. Two orbits are depicted in figures 1 and 2.

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